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Lovelock Gravity at the Crossroads of Palatini and Metric Formulations

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We consider extensions of the Einstein-Hilbert Lagrangian to a general functional of metric and Riemann curvature tensor, $\mathcal{L}(g_{\mu\nu}, R_{\mu\alpha\beta\nu})$. A given such Lagrangian describes two different theories depending on considering connection and metric (Palatini formulation), or only the metric (metric formulation) as independent dynamical degrees of freedom. Equivalence of the Palatini and metric formulations at the level of equations of motion, which as we will argue is a manifestation of the Equivalence Principle, is the physical criterion that restricts form of the Lagrangians of modified gravity theories. We prove that within the class of modified gravity theories we consider, only the Lovelock gravity satisfies this requirement.

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INTRODUCTION

General Relativity (GR) associates gravity to the (geometric) properties of space-time, metric and connection. These represent two essentially different properties of space-time, metric is the measure of length while connection defines the covariant derivative and parallel transportation. Geodesics, the curves which extremize the distance between two points, are thus specified with the metric. The worldline of a free particle, a curve along which the velocity vector is covariantly constant, is determined by the connection. For a general connection the worldline of a free particle need not be a geodesic.

In the ordinary *metric formulation* of GR, we require that the worldline of a free particle is a geodesic. This requirement fixes the connection to the Levi-Civita connection, the components of which are the Christoffel symbols. Extremizing the action with respect to the metric gives the equations of motion for the metric. In principle one can relax this requirement, treat the connection and metric as two independent fields and extremize the action with respect to both to obtain respective equations of motion. We refer to this latter treatment as the *Palatini formulation* [19]. The connection solving the equation of motion of the connection in the Palatini formulation does not necessarily coincide with the Levi-Civita connection [20]. For the Einstein-Hilbert action, however, only the Levi-Civita connection solves the corresponding equation in the Palatini formulation and the two formulations become identical [1, 2, 3].

Although successful in describing the observational and the experimental data, there exist theoretical and phenomenological motivations to study modifications or corrections to the Einstein-Hilbert action. In the theoretical side, we know that the Einstein-Hilbert action is a classical self-interacting theory. In the semi-classical regime, in principle, this action receives quantum corrections. For example string theory, as a candidate for quantum gravity, provides a framework for computing the higher order corrections to the Einstein-Hilbert action up to the field redefinition ambiguities [6]. In a phenomenological approach to cosmology and astrophysics, it has been argued that an appropriate modification of the Einstein-Hilbert action may provide an alternative resolution to dark matter and dark energy problems, and a natural framework to address the inflationary paradigm [21].

In a bottom-up approach, the general covariance imposes a weak restriction on the Lagrangian of modified gravities. Therefore, it is desirable to find additional theoretical criteria or requirements to restrict further the form of the Lagrangian. We argue that the equivalence of the Palatini and metric formulations, which is a property of the Einstein gravity theory, naturally provides such theoretical criterion and strongly restricts form of the corrections or modifications to the Einstein-Hilbert action.

Let us elaborate on the physical meaning of the equivalence of the Palatini and metric formulations. As mentioned, in the Palatini formulation a free particle does not necessarily follow a geodesic, the path which minimizes the distance. Consider a massless particle (a light ray) which should follow a path of a free particle in a given background geometry. If this path is not a geodesic, then there should exist another path, a geodesic, along which an (accelerated) object can travel faster than light. This is in contradiction with the basics of the Einstein general relativity. In another point of view, along a geodesic the particle will feel a force and hence gravity cannot be locally turned off. This is against the usual interpretation of the Equivalence Principle. In the metric formulation we do not face these contradictions. Nonetheless, in a theory of modified gravity, there is always the theoretical possibility of choosing the Palatini or metric formulations and there is no reason which one should be taken from the outset.

We take the standpoint that the "physically allowed" theories of the modified gravity are those for which the Palatini and metric formulations are (classically) equivalent. Here we consider a class of torsion-free modified

gravity theories in which the gravity part of the Lagrangian is only a functional of the Riemann tensor and metric, and not of their covariant derivatives, and the matter part of the Lagrangian does not involve the connection. We prove that within this class only the Lovelock gravity theories fulfill the requirement of equivalence of the Palatini and metric formulations.

To this end we take the following route. Deriving the equations of motion for a general Lagrangian of interest in both the metric and Palatini formulations, we first require the *consistency* of the two formulations. We implement the consistency by demanding that the Levi-Civita connection solves the equation of motion of the connection in the Palatini formulation. This makes the equation of motion for the metric identical in the both formulations. We show that only the Lovelock gravity meets the consistency requirement. We then prove the *equivalence* of the two formulations for the Lovelock gravity by considering the Lovelock theory in the Palatini formulation. We show that in the asymptotically flat space-times, only the Levi-Civita connection solves the equations of motion for the connection.

PALATINI VS. METRIC FORMULATIONS

We consider the modified gravity Lagrangians of the following form (where no explicit covariant derivative is involved):

$$S_{mod.GR} = \frac{1}{4\pi G_N}\int d^Dx\ \sqrt{-g}\ \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\alpha\beta})\,, \tag{1}$$

where the Riemann curvature tensor is defined by :

$$\begin{aligned} R^{\alpha}{}_{\beta\mu\nu} &\equiv 2\partial_{[\mu}\Gamma^{\alpha}_{\nu]\beta} + 2\Gamma^{\alpha}_{[\mu|\rho}\Gamma^{\rho}_{\nu]\beta}, \\ R_{\alpha\mu\beta\nu} &\equiv g_{\alpha\eta}R^{\eta}{}_{\mu\beta\nu}\,, \end{aligned} \tag{2}$$

and $\Gamma^{\alpha}_{\beta\nu}$ is the connection. In this work we consider the *torsion-free* theories, $\Gamma^{\alpha}_{\beta\nu} = \Gamma^{\alpha}_{\nu\beta}$. We assume that the matter part of the Lagrangian does not contain the connection. The whole action (the matter plus the gravity parts) can be understood either in the *metric formulation* where the connection is the Levi-Civita connection, or in the *Palatini formulation* where the connection is an independent variable and does not necessarily coincide with the Levi-Civita.

Let us derive the equations of motion of this action in both the metric and Palatini formulations. The equations of motion in the metric formulation are [8]

$$\Gamma^{\mu}_{\alpha\beta} = \{{}^{\mu}_{\alpha\beta}\} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta})\,, \tag{3a}$$

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} + \frac{1}{2}\mathcal{L}g^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial R_{\mu\rho\alpha\beta}}R^{\nu}{}_{\rho\alpha\beta} + 2\nabla_{\{\alpha}\nabla_{\beta\}}\frac{\partial\mathcal{L}}{\partial R_{\mu\alpha\beta\nu}} = -T^{\mu\nu}, \tag{3b}$$

where $\{{}^{\mu}_{\alpha\beta}\}$ is the Christoffel symbol, while in the Palatini formulation the equations of motion for the connection and the metric read

$$\nabla_\nu(\sqrt{-g}\frac{\partial\mathcal{L}}{\partial R_{\mu\{\alpha\beta\}\nu}}g_{\mu\rho}) = 0, \tag{4a}$$

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} + \frac{1}{2}\mathcal{L}g^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial R_{\mu\rho\alpha\beta}}R^{\nu}{}_{\rho\alpha\beta} = -T^{\mu\nu}, \tag{4b}$$

where $T^{\mu\nu}$ in the r.h.s of (3b) and (4b) stands for the energy-momentum tensor of the matter field. Note that the partial derivatives of $\mathcal{L}$ are taken assuming that $g_{\mu\nu}$ and $R_{\mu\nu\alpha\beta}$ are independent variables, and the partial derivative coefficients are uniquely fixed to have precisely the same tensor symmetries as the varied quantities. For a general Lagrangian, (3) and (4) are not equivalent [20] or even consistent.

REQUIRING THE CONSISTENCY

Requiring the consistency of the two formulations amounts to demanding (3a) or equivalently,

$$\nabla_\alpha g_{\mu\nu} = 0\,, \tag{5}$$

to be a solution of (4a). Expand (4a) as follows

$$\frac{\partial\mathcal{L}}{\partial R_{\mu\{\alpha\beta\}\nu}}\nabla_\nu(\sqrt{-g}g_{\mu\rho}) + \sqrt{-g}g_{\mu\rho}\nabla_\nu(\frac{\partial\mathcal{L}}{\partial R_{\mu\{\alpha\beta\}\nu}}) \ = 0\,. \tag{6}$$

When (5) holds the first term of (6) vanishes yielding

$$\nabla_\nu(\frac{\partial\mathcal{L}}{\partial R_{\mu\{\alpha\beta\}\nu}}) = 0\,. \tag{7}$$

Now let $\mathcal{L}$ have a Taylor expansion in terms of the Riemann tensor. For Lagrangians of our interest the expansion coefficients are functionals of metric only and when (5) holds these coefficients are covariantly constant. Therefore, in our case we can use the "chain rule with the covariant derivative" on (7) to obtain

$$\frac{\partial^2\mathcal{L}}{\partial R_{\mu\{\nu\beta\}\alpha}\partial R_{\rho\sigma\lambda\gamma}}\nabla_\alpha R_{\rho\sigma\lambda\gamma} = 0. \tag{8}$$

Recalling the Bianchi identity $\nabla_{[\alpha}R_{\rho\sigma]\lambda\gamma} = 0$, (8) is guaranteed to be satisfied if [22]

$$\frac{\partial^2\mathcal{L}}{\partial R_{\mu\{\nu\beta\}\alpha}\partial R_{\lambda\gamma\rho\sigma}} = \frac{\partial^2\mathcal{L}}{\partial R_{\mu\{\nu\beta\}\sigma}\partial R_{\lambda\gamma\alpha\rho}} = \frac{\partial^2\mathcal{L}}{\partial R_{\mu\{\nu\beta\}\rho}\partial R_{\lambda\gamma\sigma\alpha}}, \tag{9}$$

In what follows we show that only the Lovelock gravity Lagrangian satisfy the Palatini-metric consistency requirement, summarized in (9). Let us, however, first briefly review the Lovelock theory. David Lovelock used the following assumptions to restrict form of the action for pure gravity including the higher derivative terms [9]:

1. The generalization of the Einstein tensor, hereafter denoted by $A_{\mu\nu}$, should be a symmetric tensor of rank two; $A_{\mu\nu} = A_{\nu\mu}$,

2. $A_{\mu\nu}$ is concomitant of the metric and its first two derivatives, $A_{\mu\nu} = A_{\mu\nu}(g, \partial g, \partial^2 g)$,

3. $A_{\mu\nu}$ is divergence free, $\nabla^\mu A_{\mu\nu} = 0$.

(We should stress that Lovelock was working in the "metric formulation" assuming $\nabla_\alpha g_{\mu\nu} = 0$.) In a series of theorems [10], Lovelock proved that the above three assumptions are fulfilled for the generalized Einstein tensor derived only from the following Lagrangian density [12]

$$\mathcal{L}_{Lovelock} = \sum_{n=0}^{[\frac{D}{2}]} a_n\, \theta^{\mu_1\cdots\mu_{2n}\nu_1\cdots\nu_{2n}}(g) \prod_{p=1}^{n} R_{\mu_{2p-1}\mu_{2p}\nu_{2p-1}\nu_{2p}}\,, \tag{10}$$

where D is the number of the dimensions of space-time and $[\frac{D}{2}]$ represents the integer part of $\frac{D}{2}$, and

$$\theta^{\mu_1\cdots\mu_{2n}\nu_1\cdots\nu_{2n}}(g) \;=\; \det \begin{vmatrix} g^{\mu_1\nu_1} & \cdots & g^{\mu_{2n}\nu_1} \\ \vdots & & \vdots \\ g^{\mu_1\nu_{2n}} & \cdots & g^{\mu_{2n}\nu_{2n}} \end{vmatrix} \tag{11}$$

a_n's are some constant values of proper dimensionality. We refer to the n^{th} term in (10) as the n^{th} order Lovelock gravity. At its zeroth and first order, Lovelock gravity coincides respectively with the cosmological constant and the Einstein-Hilbert action. Its second order coincides with the Gauss-Bonnet Lagrangian density. The compact form of its higher orders becomes more involved, e.g. see [13, 14] for the explicit form of the third and fourth orders [23].

In order to prove that the consistency of the Palatini and metric formulations happens only for the Lovelock gravities, we first show that when the consistency holds the generalized Einstein tensor satisfies the Lovelock's assumptions. The generalized Einstein tensor is defined by the variation of the action with respect to metric, that is

$$A^{\mu\nu}_{\text{metric}} \equiv \text{The l.h.s of the Eq. (3b)}\,, \tag{12a}$$
$$A^{\mu\nu}_{\text{Palatini}} \equiv \text{The l.h.s of the Eq. (4b)}\,, \tag{12b}$$

where the subscripts indicates the formulation. The Palatini-metric consistency requirement implies that $A^{\mu\nu}_{\text{metric}} = A^{\mu\nu}_{\text{Palatini}}$, which using their explicit forms presented in (3b) and (4b), can be expressed as

$$A^{\mu\nu}_{\text{metric}} - A^{\mu\nu}_{\text{Palatini}} = 2\nabla_{\{\alpha}\nabla_{\beta\}} \frac{\partial \mathcal{L}}{\partial R_{\mu\alpha\beta\nu}} = 0\,. \tag{13}$$

Assuming (5) and (7), and recalling the symmetries of the Riemann tensor, one can immediately verify that the above equation holds. The remaining step is to recall that the Riemann tensor –due to (5)– contains at most the second derivative of the metric and $A^{\mu\nu}_{Palatini}$, by definition, involves the powers of the Riemann tensor and not its derivatives. Therefore, the generalized Einstein tensor being concomitant of the metric and its first two derivatives fulfills the second Lovelock assumption. Since we have derived the Einstein tensor from a generally covariant Lagrangian, the first and third Lovelock assumptions hold too. The uniqueness theorems of Lovelock [10] then proves that the consistency of the metric and Palatini formulation can hold only for the Lovelock gravities.

In order to complete the proof of the consistency, we must verify that for Lovelock Lagrangians, (5) solves either the equation of motion for the connection (4a), or equivalently (9). This verification is obvious if we insert (10) into (9) and recall the invariance of the determinant (11) under the cyclic permutation of its three rows. (Lovelock Lagrangians satisfy (9) without symmetrization over β and ν indices.)

THE EQUIVALENCE OF THE FORMULATIONS

To argue for the equivalence, we should show that the only solution to (4a) for $\mathcal{L} = \mathcal{L}_{Lovelock}$ is the Levi-Civita connection. To this end, we notice that when $\nabla_\alpha g_{\mu\nu} \neq 0$, generically $R_{\mu\nu\eta\gamma} \neq R_{\eta\gamma\mu\nu}$. However, we note that

for a general connection, the Lovelock Lagrangians are only functional of the part of the Riemann tensor which satisfies $R_{\nu\mu\eta\gamma} = -R_{\mu\nu\eta\gamma}$ and $R_{\mu\nu\eta\gamma} = R_{\eta\gamma\mu\nu}$.

This follows from the definition of the Lovelock lagrangian densities in (10), and the antisymmetric property of the determinant (11) under permutation of its two rows or columns, and the Bianchi identity of $R_{\mu[\nu\eta\gamma]} = 0$.

The above lemma then implies that the Lovelock Lagrangians satisfy (8) and (9), and subsequently (7), for a general connection. Therefore, the equation of motion of the connection in the Palatini formulation (6) takes the form

$$\frac{\partial \mathcal{L}_{\text{Lovelock}}}{\partial R_{\mu\{\alpha\beta\}\nu}} \frac{\nabla_\nu(\sqrt{-g}g_{\mu\rho})}{\sqrt{-g}} + g_{\mu\rho}\frac{\partial^2 \mathcal{L}_{\text{Lovelock}}}{\partial R_{\mu\{\alpha\beta\}\nu}\partial g_{ab}} \nabla_\nu g_{ab} = 0\,. \tag{14}$$

For the first order Lovelock Lagrangian density, (14) is an algebraic equation for the connection whose unique solution is the Levi-Cevita connection (see *e.g.* section 3.4 of [5]). For a general Lovelock Lagrangian density in the Palatini formulation, however, (14) is a first order differential equation for the connection. If the connection coincides with the Levi-Cevita connection in a single point on a regular and connected space-time manifold then the uniqueness theorems of the solutions to the differential equations guarantee that the connection is the Levi-Cevita connection globally.

In an asymptotically flat space-time, the Riemann tensor vanishes in the asymptotic infinity. Since both of the Riemann tensor and the torsion vanish in the asymptotic infinity then we can choose coordinates in such a way that the connection coincides with the Levi-Cevita connection in the point of the asymptotic infinity. Therefore, in the asymptotically flat space-times, the equivalence of the Palatini and metric formulation for the Lovelock gravity is guaranteed.

SUMMARY AND OUTLOOK

We have discussed that a strict interpretation of the Equivalence Principle, in the absence of torsion, requires the connection to be the Levi-Civita connection. When the Lagrangian is not a functional of covariant derivatives of the curvature or metric, this requirement implies *the consistency of Palatini and metric formulations* and restricts the Lagrangians only to the Lovelock gravity [24].

The requirement of equivalence or consistency of the Palatini and metric formulations can be imposed on more general theories than those we considered here. For example one may use this requirement to restrict the form of action for gravity when torsion does not vanish, or when the Lagrangian involves the covariant derivatives of Riemann or metric. It can also be used to restrict form of the non-minimal coupling between matters and gravity [15].

The equivalence of the Palatini and metric formulations can also serve as a criterion for fixing the field redefinition ambiguities arising in the string loop or worldsheet corrections to supergravities [6]. The proposals for fixing the field redefinition ambiguities include the MM-criterion [16] and the ghost-free condition [17]. Noting that Lovelock Lagrangians are ghost-free [18], the Palatini-metric equivalence criterion is in agreement with the ghost-free criterion. Investigating the Palatini-metric equivalence criterion for matter fields non-minimally coupled to gravity, however, precedes its comparison to the MM-criterion.

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[19] There exists a debate in the literature on calling this after Palatini [1] or Einstein [2]. Ref. [3] argues that it should be called after Einstein.

[20] The inequivalence of the metric and Palatini (also known as metric-affine, first order or mixed) formulations for a general Lagrangian is a well studied issue in the literature; *e.g.* [4].

[21] Implications of $f(R)$ modified gravity theories on cosmology, and astro and solar system physics bas been studied extensively in the both Palatini and metric formulations; *e.g.* [7].

[22] Note that with the Levi-Civita connection, *i.e.* when (5) holds, the Riemann tensor has its usual symmetries on its indices, *e.g.* $R_{\lambda\gamma\rho\sigma} = R_{\rho\sigma\lambda\gamma}$.

[23] Note that except for $n = 1$, none of the Lovelock gravities are of the form of $\mathcal{L} = \mathcal{L}(R, R_{\mu\nu})$.

[24] Similarities between the Einstein-Hilbert action and the Lovelock gravity have also been noted in [11].